Direct Numerical Simulation and Large Eddy Simulation of Forced Turbulence in 1-D Burgers' Equation Using Different SGS Models

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Abstract

Direct numerical simulation and large eddy simulation of forced turbulence in 1-D Burgers' equation using different SGS models have been investigated in this project. Using pseudo spectral methods, 1-D Burger's equation have been solved and results of various proposed models have been compared with DNS results. It is also prooved analytically that the Bardina model with sharp cut-off filter generates results as same as Smagorinsky model. Different featues of LES has been investigated and parameters such as flatness, skewness, Fourier transform of the filter and etc. has been studied.

Keywords: DNS, LES, SGS models, Burgers' equation

1 Governing equation: Burger's equation

In this study, forced Burger's equation in one spatial dimension had been investigated numerically. This equation is as follows.

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial t}(\frac{u^2}{2}) = \nu \frac{\partial^2 u}{\partial x^2} \tag{1}$$

Multidimensional analogues of Burger's equation were originally proposed as simplified models of Navier-Stokes (N-S) turbulence. By design, Burger's equation is the simplest model of hydrodynamic flow that captures the interaction of nonlinear wave propagation and viscosity. Hence, there exists a strong analogy between the compressible N-S equation and Burger's equation. Both systems exhibit shock wave formations that dissipate energy at small scales. Burger's turbulence is thus often viewed as a pared-down model of acoustic turbulence [1]. Burger's equation shares the following properties with the NavierStokes equation:

1. same type of advective nonlinearity
2. presence of a diffusion term from which a Reynolds number may be defined

3. many invariance and conservation laws in common: invariance under translations in space and time, parity invariance, conservation of momentum and energy.

Indeed, an important property of the NavierStokes equation, not shared by the Burger's equation, is the sensitivity to small changes in the initial conditions in the presence of boundaries or driving forces and at sufficiently high Reynolds numbers. Hence, the Burger's equation is not a good model for one of the most important aspects of turbulence: the spontaneous generation of randomness by chaotic dynamics [2].

2 Direct Numerical Simulation of Burgers' equation

The differential form of the Burger's equation is firstly solved in a DNS approach in order to build a database for comparison. By definition, direct numerical simulation (DNS) is the numerical solution of the NavierStokes equations without recourse to modeling. In concept, fluid motions are resolved down to the Kolmogorov length scale. Kolmogorov theory predicts the ratio of the integral scale to the Kolmogorov scale to be on the order of $Re^{3/4}$, where Re is the Reynolds number based upon the integral scale. In three spatial dimensions and time, the computational requirements of DNS scale as Re^3. Consequently, for the high-Reynolds-number flows of engineering interest, the computational requirements of fully resolved DNS are staggering [3].

Spectral and pseudo spectral methods have become popular in approximating solutions of advection equations arising in many sciences. In this project spectral methods was employed to solve 1-D Burger's equation. Furthermore, by use of the fast Fourier transform (FFT) algorithms, implementation of explicit pseudo spectral schemes can be accomplished economically [4].

The initial condition will be imposed by injecting energy to some low magnitude wave numbrers permanently. It was found that amount of energy injection and also number of wave numbers which are injected, had a great influence on the results. For instance, by injecting energy of magnitude 1 to k=3 and 5 some fluctuations in energy spectrum were seen. In order to eliminate these fluctuations, two solution were found:

1. Reducing the amount of injected energy to 0.1.

2. Injecting energy to k=6, 8, 9. (Mr. Amani's contribution)

Using proposed solutions, fluctuations was reduced greatly. However these fluctuations are unavoidable and also were seen in results of [5, 6]. It shoud be noted that periodic boundary condition was imposed in a domain width of 2π.

The other important issue is computational cost of the problem. So to reduce the CPU time and also to remove aliasing error, Padding method was

used to compute Fourier transform of convective terms. By the aid of this method, computational cost of the problem is reduced from $O(N^2)$ algebric calculations to $O(NlogN)$ calculations. To verify this fact, Fig.1 shows variation of the CPU time versus $NLogN$. It is obvious that the relation is linear mainly.

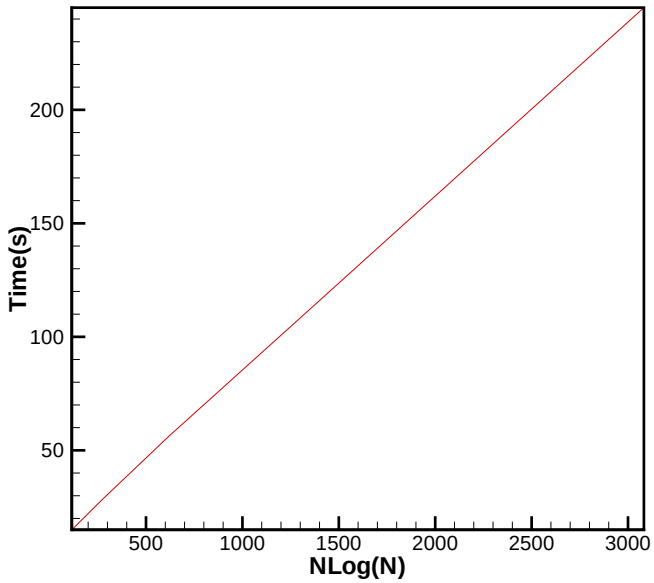

Figure 1: Variation of CPU time with respect to the number of nodes

It is also important that, at what time our solution will be stationary or steady state. Results of my code showed that the steady state was reached at t=6s, approximately with error of 10^{-4}. This time of steadiness is to some extant the same as the values reported by [7].

2.1 Results of DNS

To test the accuracy of our numerical method, we have performed a priori tests using data from a DNS at high mesh points. A standard pseudo-spectral direct numerical simulation of the Burger's equation was carried out on a grid with different number of mesh points. A Second-order time accurate explicit Adams-Bashforth differentiating scheme was used to integrate the nonlinear term. Partial dealiasing was achieved through the use of Padding algorithm which was explained previously in the class.

Fig. 2 shows the steady state energy spectrum for various number of nodes and two different viscosities. A time step of δt =0.0001 has been used to solve Burger's equtiaon. Clearly, up to N=256 and 2048 for ν= 0.1 and 0.01 respectively, the direct simulation without the model is not able to capture the physics correctly, as the energy is not dissipated enough at high wave

numbers, and is reflected back towards the larger scales to form a tail in energy spectrum. As expected by increasing Reynolds number or equivalently decreasing viscosity, piling up occures at higher wave numbers. It can be seen that the inertial sub-range evolves toward the correct slope -2.

Two important statistical parameters in turbulence which had been considered in this study are skewness and flatness. The definition of skewness and flatness are as follows:

$$S = \frac{\langle (\frac{\partial u}{\partial x})^3 \rangle}{(\langle \frac{\partial u}{\partial x})^2 \rangle^{3/2}} \tag{2}$$

$$F = \frac{\langle (\frac{\partial u}{\partial x})^4 \rangle}{(\langle \frac{\partial u}{\partial x})^2 \rangle^2} \tag{3}$$

Fig.3 illustrates the velocity derivative skewness and flatness for ν=0.1 and ν=0.01. The main aspect of this diagrams is the negative value of the velocity derivative skewness varying around -3.9 and -1.5 for ν=0.1 and 0.01 respectively. It is seen from this figure that for ν=0.1, the time variation of skewness is approximately intermittent and non-Gaussian. But by increasing viscosity intermittency of skewness diagram vanishes. From Fig.3 it is obvious that the mean value of flatness is very high. This fact demonstrates the intermittency of velocity derivative signal which is an expectable reality. Negative skewness could be also interpreted by means of gas dynamic concepts. But inspite of great effort I could not find any useful text about this matter.

3 Larg Eddy Simulation of Burger's equation

A promising alternative to DNS is large-eddy simulation (LES). In LES, large-scale fluid motions are resolved in space and time on a suitable computational grid; however, the effects of unresolved scales of motion upon the larger scales are modeled. For pseudo-spectral methods the problem is that one cannot avoid introducing dissipation even at scales greater than the grid size. The reason for this is a numerical instability which manifests itself in stagnation of energy in small scales and called bottle-neck instability. To dissipate this energy, a smoothly increasing with wave number dissipation function has to be introduced. To achieve high Reynolds numbers such a dissipation function can be chosen to be of a hyperviscosity type, i.e. a function which is steeper than the real viscosity, but it cannot be made too steep to avoid the bottle-neck instability [8].

Relative to DNS, LES is conducted on relatively coarse grids at reasonable computational expense. Filtering the governing equations generates residual stresses that require closure either by modeling or approximation. Relatively recent advances such as dynamic modeling and deconvolution methods have made LES practical for application to certain flows of engineering interest [3]. in order to employ different SGS models and investigate their power to model

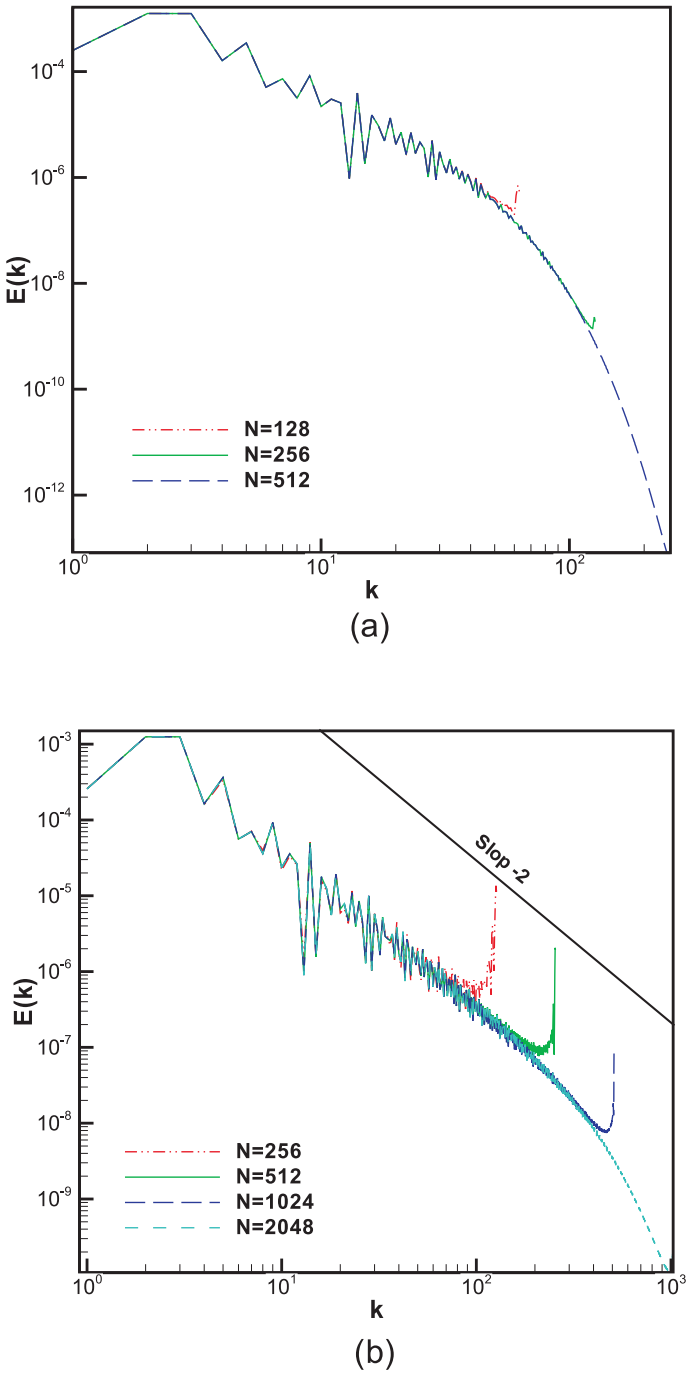

Figure 2: Direct numerical simulation of Burger's equation with different number of nodes, (a) ν=0.1, (b) ν=0.01

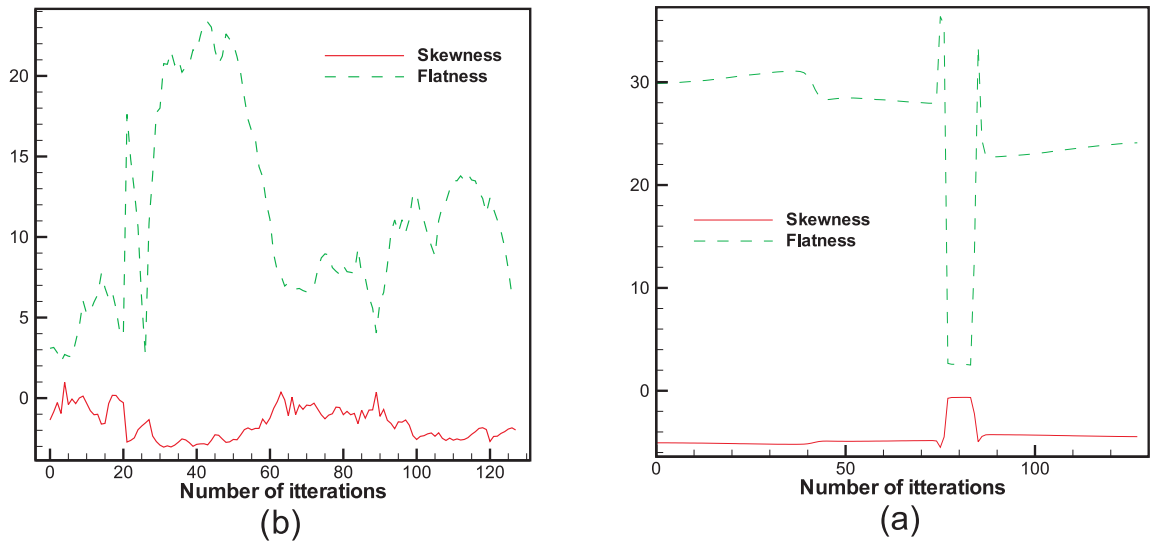

Figure 3: Skeness and flatness of velocity derivative (a) ν=0.1, (b) ν=0.01

low scale structures, a filtering operation is imposed on the Burger's equation. Then filtered Burger's equation will be:

$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial}{\partial t}(\frac{\bar{u}^2}{2}) = \nu \frac{\partial^2 \bar{u}}{\partial x^2} - \frac{\partial \tau}{\partial x} \tag{4}$$

$$\tau = \frac{1}{2}(\overline{u^2} - \bar{u}^2) \tag{5}$$

3.1 Smagorinsky model

The Smagorinsky eddy viscosity model approximate the subgrid scale stress according to the traditional Smagorinsky formula.

$$\tau = -(c_s \Delta)^2 \left| \frac{\partial \bar{u}}{\partial x} \right| \frac{\partial \bar{u}}{\partial x} \tag{6}$$

where Δ is a characteristic length which is taken to be proportional to the filter width and C_s is a constant to be properly tuned. It is obvious that due to the positive eddy viscosity coefficient there is no back scatter and energy transfer everywhere is from the filtered motions to the residual motions. Results of our numerical code was explored in order to find an optimum value of C_s in which lack of energy dispersion at high wave numbers is avoided. Optimum valus of smagorinsky coefficient for different viscosities are listed in table 1.

As mentioned in the class, $< \bar{S}\bar{S} >$ which is a function of $2\nu k^2 E(k)$ gets the greatest influence from the structures around cut of wave number. Fig. 4

Table 1: Optimum valus of smagorinsky coefficient for different viscosities

ν	C_s
0.1	0.15
0.01	0.3
0.001	0.38
0.0001	0.45

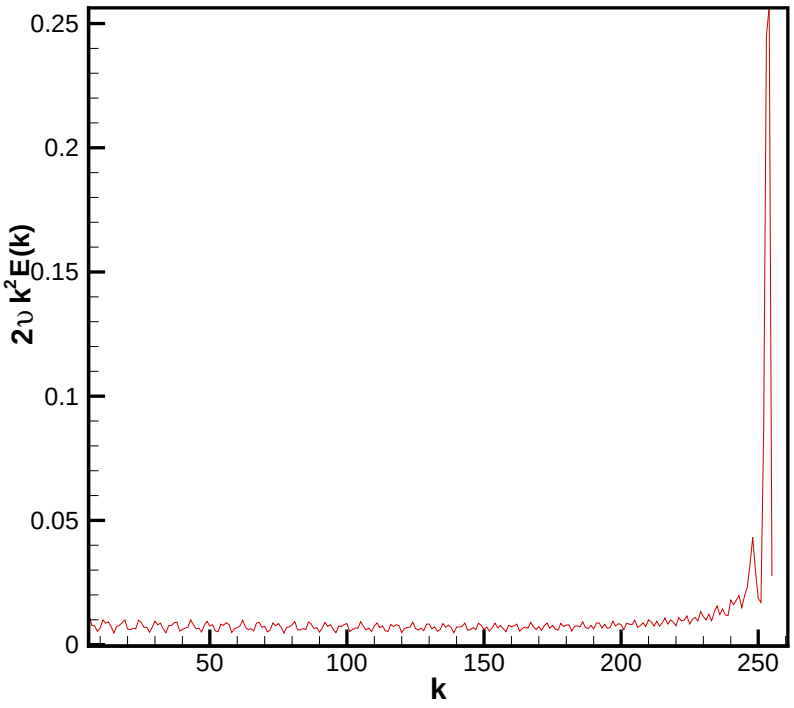

Figure 4: Variation of $2\nu k^2 E(k)$ with respect to the wave numbers for ν=0.01

presents variation of $2\nu k^2 E(k)$ with respect to the wave numbers for ν=0.01 and N=512. As it is obvious $2\nu k^2 E(k)$ has a peak around cut off wave number.

Energy spectrum resulted from Smagorinsky model with different coefficients for ν=0.1 and ν=0.01 are shown in Fig. 5. From these figures it is evident that by increasing C_s, energy dissipation in high wave numbers increases until an optimum value beyond which due to the extra dissipation, the curve of energy spectrum turns downward. So an optimum value could be distinguished easily from these figrues.

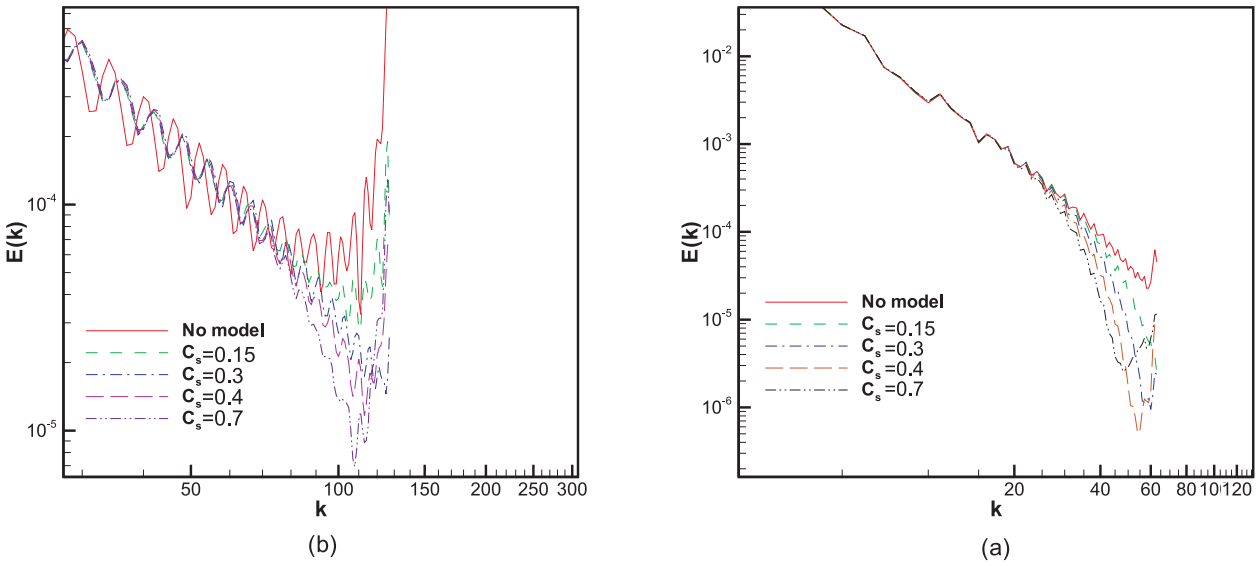

Figure 5: Energy spectrum resulted from Smagorinsky model with different coefficients (a) ν=0.1, (b)ν=0.01

To investigate the efficiency of Smagorinsky model, the Smagorinsky model with optimum coefficient was compared with the DNS results and as well as results with no model for ν=0.1 and ν=0.01. A tail in energy spectrum was seen for the curves without model which shows the well known energy pile-up close the cut-off frequency. But results of Smagorinsky model is coincide with the DNS results. Number of mesh points is 128 for all LES results.

The main idea of LES is to simulate high Reynolds flow fields on coarse meshes. After verifying Smagorinsky model with DNS results, amount of viscosity is decreased to 0.0001 and energy spectrum for different C_s is plotted in Fig. 7. As expected results with 256 mesh point and no model applied, are so terrible. However by applying Smagorinsky model dissipation is also added in high wave numbers and so energy spectrum curve is improved considerably.

One of the most important filters is sharp cut-off filter which is used greatly by many researcher. In this study sharp cut-off filter is also used implicitly to filter the Burger's equation. After getting results of LES, now we are able to

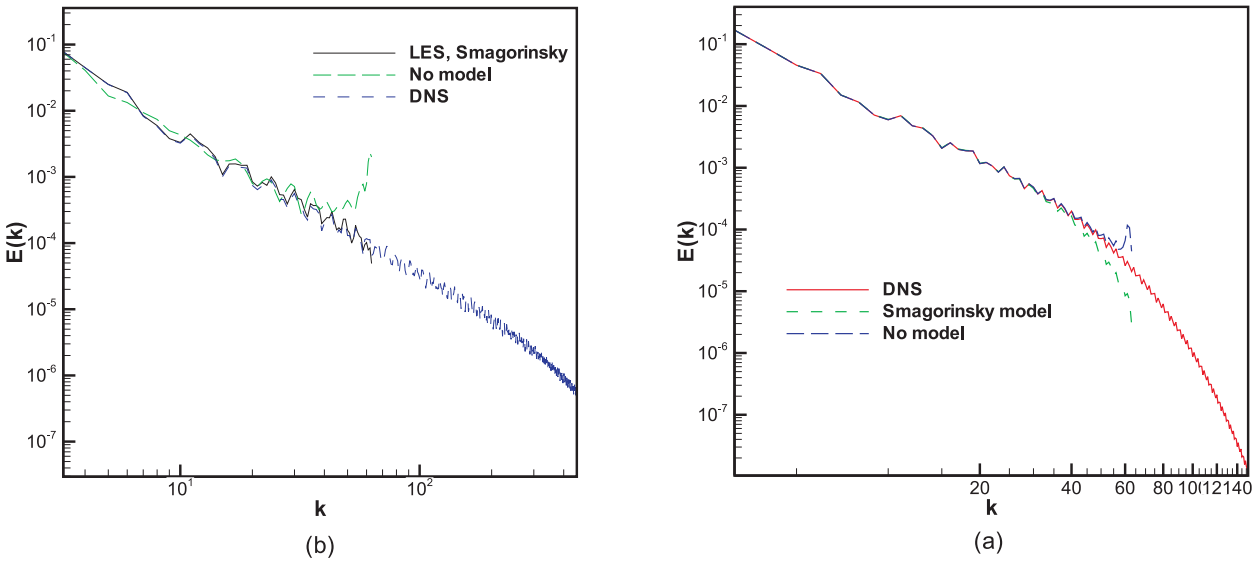

Figure 6: Energy spectrum resulted from Smagorinsky model with optimum coefficient compared with DNS (a) ν=0.1, (b)ν=0.01

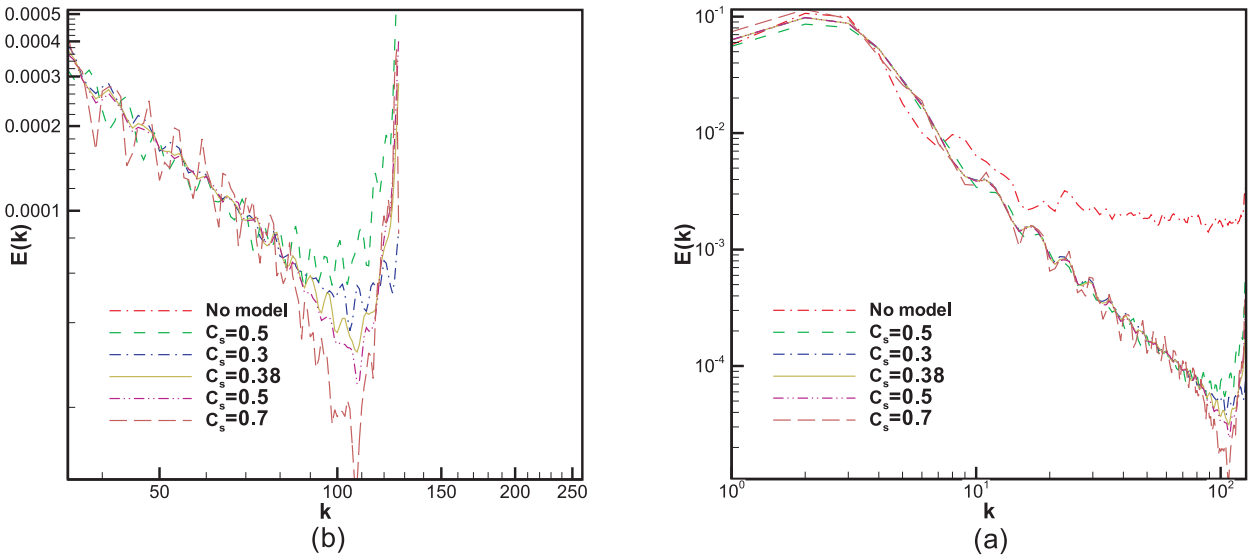

Figure 7: Energy spectrum resulted from Smagorinsky model with optimum coefficient (a) ν=0.0001, (b) a close snapshot

compute the real filter operator based on Eq. 7.

$$E_{LES} = \widehat{G(k)}^2 E_{DNS} \tag{7}$$

The $\widehat{G(k)}^2$ computed for ν=0.1 and ν=0.01 are sketched in Fig. 8. It was expected to reach a heavyside function for Fourier transform of the filter. But as it is obvious from Fig. 8, our numerical results deviate from the heavyside function to some extant. This observation stems from the existence of viscous or diffusive term in the Burger's equation.

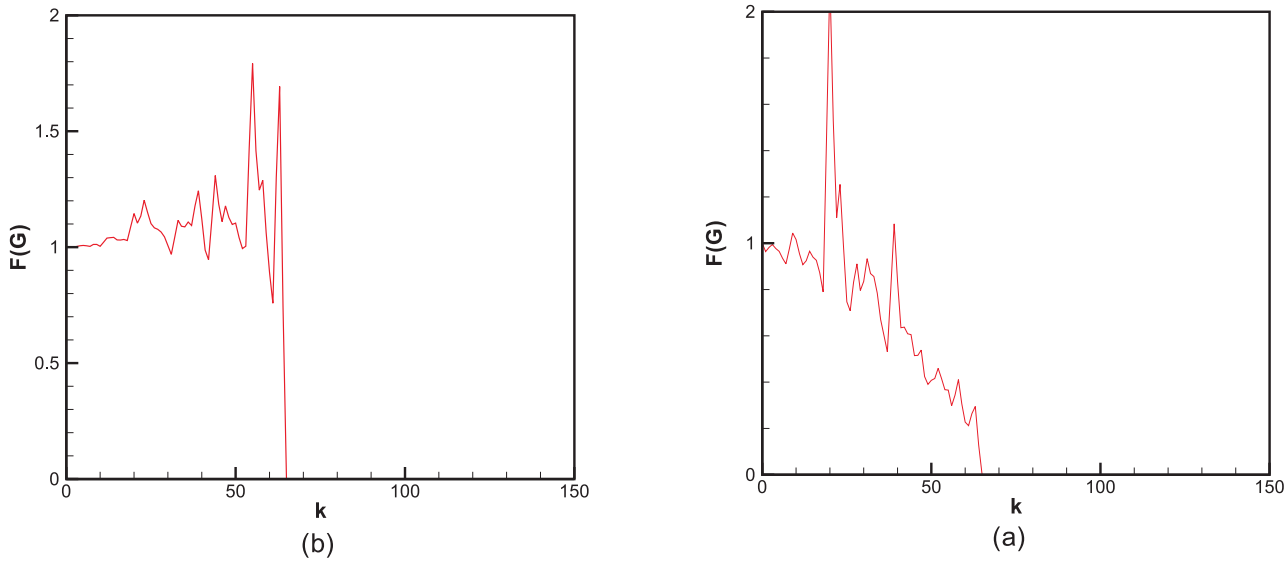

Figure 8: Fourier transform of sharp cut off filter employed to filter Burger's equation (a) ν=0.1, (b)ν=0.01

3.2 Dynamic Smagorinsky model

It is well known that, thanks to the Germano identity, the constant in Eq. 4 can be determined in a dynamic way without any ad hoc tuning. Indeed, by choosing a second characteristic length $\hat{\Delta} = 2\Delta$ and by applying again a filtering of this width on to Eq. 4 one gets

$$L = T - \hat{\tau} = \frac{1}{2}(\widehat{\bar{u}^2} - \hat{\bar{u}}^2) \tag{8}$$

Using Smagorinsky model to estimate T and τ and defining M, one can write:

$$M = \hat{\Delta}^2 \left|\frac{\partial \hat{\bar{u}}}{\partial x}\right| \frac{\partial \hat{\bar{u}}}{\partial x} - \Delta \left|\widehat{\frac{\partial \bar{u}}{\partial x}}\right| \frac{\partial \bar{u}}{\partial x} \tag{9}$$

Now by minimizing error we can find a prpoper value of C_s from

$$C_s^2 = -\frac{< LM >}{< MM >} \tag{10}$$

Ensemble averages in above formula is interpreted as a spatial averaging and is added to find a uniqe value for whole field and also to avoid fluctuations of C_s. Inspite of its such great advantages as independency from tuning an undetermined constant, dynamic Smagorinsky model has two disadvantages. The first one is high computational cost specially in great scale problems and the other one is approximations in determination of C_s. It should be noted that in this study clipping is performed by setting negative values of C_s to zero, so the backscatter is eliminated.

An interesting feature to analyse is the time behaviour of the dynamic Smagorinsky constant. The ensemble average values of C_s as a function of number of iterations is illustrated in Fig. 9. To verify the results of this section in table 2, calculated coefficients with dynamic model is compared with the coefficients of table 1 obtained from static Smagorinsky model. Coefficients listed in tabel 2 are averaged over time to be comparable.

Table 2: Comparison of computed coefficient of dyanmic Smagorinsky method with "Static" Smagorinsky method

ν	C_s from Static Smagorinsky	C_s from dynamic Smagorinsky
0.1	0.15	0.35
0.01	0.3	0.26
0.001	0.38	0.28
0.0001	0.45	0.26

Fig. 10-a shows the energy spectrum obtained from static and dynamic Smagorinsky model for ν=0.01 and compares them with an exact DNS spectrum and Fig. 10-b illustrates the energy spectrum obtained from static and dynamic Smagorinsky model for ν=0.0001 and compares it with results obtained when no model was applied. In both cases good spectral resolution of dynamic and static Smagorinsky model with eachother as well as with DNS spectrum was seen.

3.3 Structure function model

The structure function model for 1-D case is obtained from averaging the square of the difference in velocity over two neighboring mesh points. Similar to the Smagorinsky model, structure function model also needs to determine a coefficient which in this study is called C_{sf}. So an optimum value of C_{sf} must be determined so that results of DNS and LES coincide. From Fig. 11

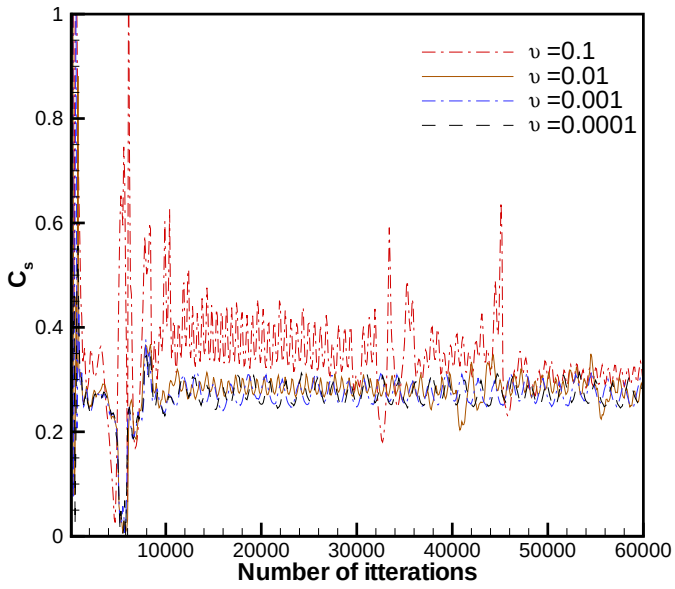

Figure 9: Values of computed dynamic Smagorinsky model coefficient,

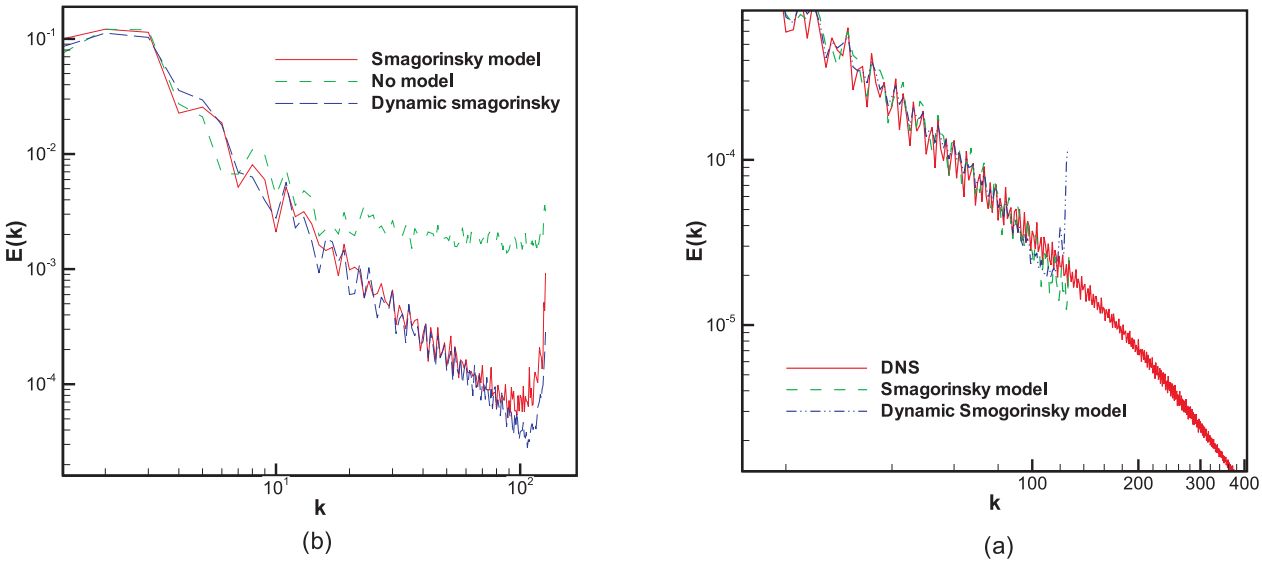

Figure 10: Comparison of energy spectrum obtained using Dynamic Smagorinsky or Smagorinsky model with results of DNS or with no model, (a) ν=0.01, (b) ν=0.0001

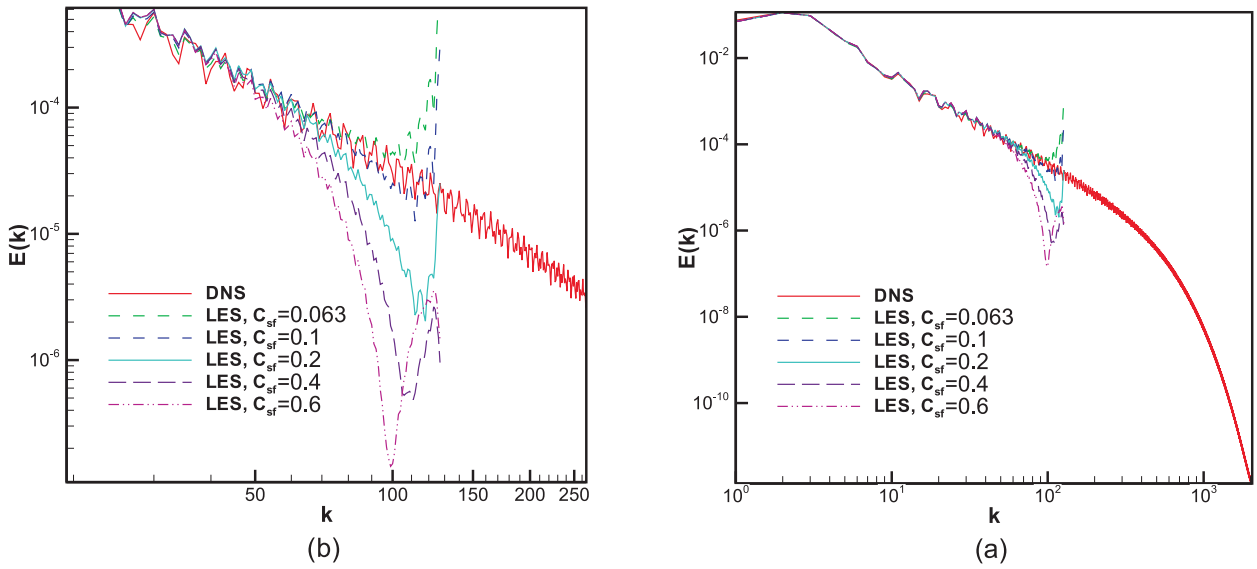

Figure 11: (a) Energy spectrum for different coefficients of structure function model comparing with DNS (b) a close snapshot, $\nu = 0.01$

the optimum value of C_{sf} has been found to be 0.1 for ν=0.01. For coefficients less than or more than 0.1 this model is less or more dissipative respectively.

Similar to the other models, in Fig. 12 viscosity is reduced to 0.0001 for 256 mesh points and energy spectrum is plotted comparing with results obtained from solving the equation with no model. It is seen that Smagorinsky model and structure function model are very close to eachother with great improvement relative to the results obtained with no specific model.

3.4 Bardina model

Bardina model as a scale-similarity model is investigated in this section. As discussed in the class, the residual stress can be decomposed in different ways to Leonard stress, the cross stress and the SGS Reynolds stress. Two latter stresses have been modeled by Smagorinsky model and Leonard stress has been directly evaluated. But after applying this new model, amazingly it was recognized that the results of current model and Smagorinsky model are nearly the same. Then I statrted to find the reason of this fact. Afetr some effort I understood that according to Eqs. 11 and 12 the Fourier transform of Leonard stress for sharp cut-off filter is analytically zero.

$$\widehat{L^0} = F(L^0) = F(\overline{\overline{u}.\overline{u}} - \bar{\bar{u}}.\bar{\bar{u}}) \tag{11}$$

for sharp cut off filter

$$\widehat{L^0} \quad = \quad F(L^0) = F(\overline{\overline{u}.\overline{u}} - \bar{u}.\bar{u}) = F(\overline{\overline{u}.\overline{u}}) - F(\bar{u}.\bar{u})$$

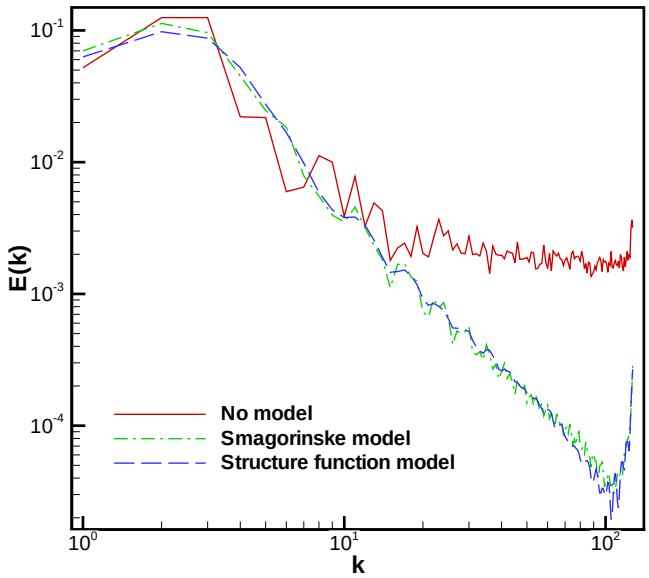

Figure 12: Comparison of energy spectrum for ν=0.0001 with and without structure function model and Smagorinsky model

$$\begin{aligned} &= H(k-k_c)F(\bar{u}.\bar{u}) - F(\bar{u}.\bar{u}) \\ &= \begin{cases} F(\bar{u}.\bar{u}) - F(\bar{u}.\bar{u}) & k < k_c \\ -F(\bar{u}.\bar{u}) & k > k_c \end{cases} = 0 \end{aligned} \tag{12}$$

So as it is shown in Fig. 13, results of Bardina model and Smagorinsky model are nearly the same.

4 Conclusions

Direct numerical simulation and large eddy simulation of forced turbulence in 1-D Burgers' equation using different SGS models have been investigated in this project. Using pseudo spectral methods, 1-D Burgers' equation have been solved and results of various proposed models have been compared with DNS results. Smagorinsky model and structure function has been used as an undetermined coefficient models which implies inconvenience for researchers to employ them. So dynamic Smagorinsky model as an example of dynamic models has been used in Forced turbulence Burger's equation. It is also prooved analytically that the Bardina model with sharp cut-off filter generates results as same as Smagorinsky model.

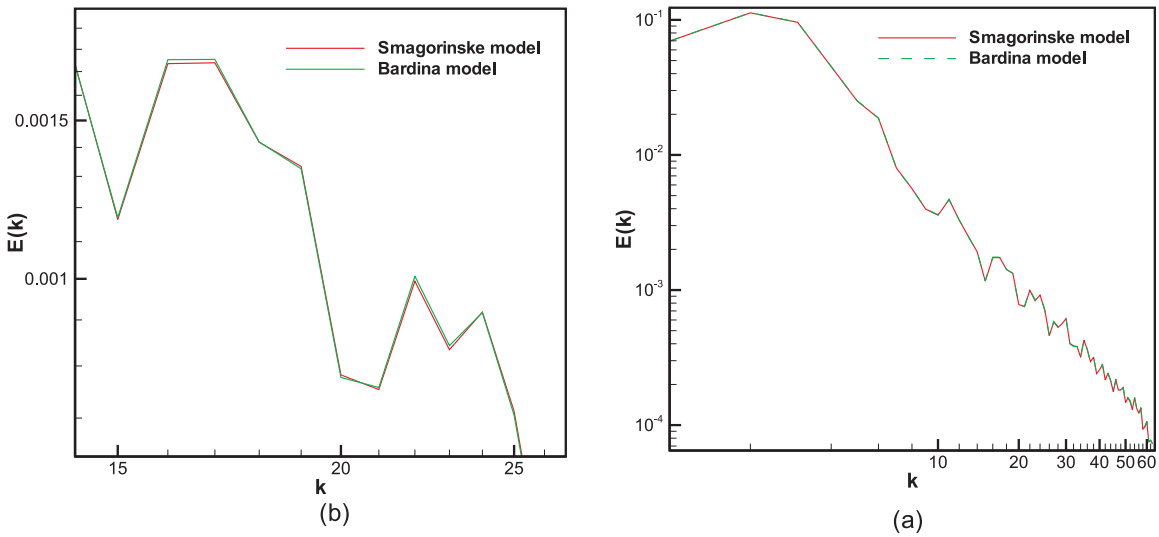

Figure 13: (a) Energy spectrum with ν=0.1 using Bardina model, (b) a close snapshot

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